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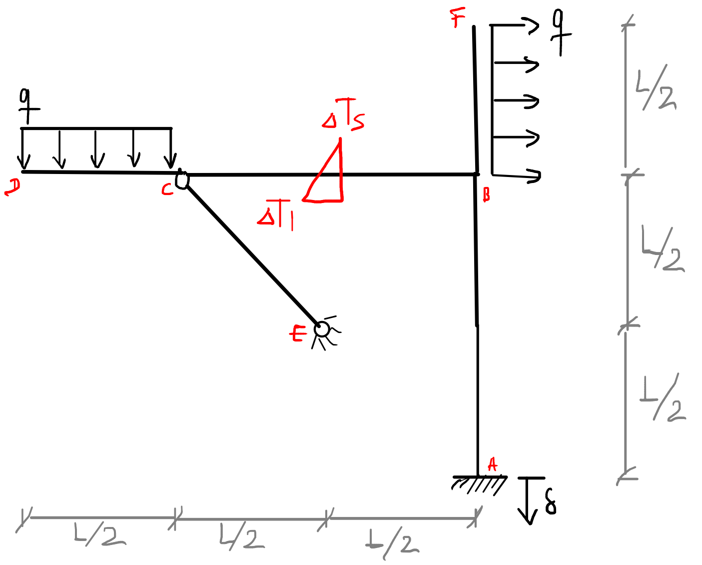
DIPARTIMENTO DI ARCHITETTURA, INGEGNERIA DELLE COSTRUZIONI E AMBIENTE COSTRUITO

Find the internal actions N, V and M for the presented structure.



$\Delta T_i = 30°C$; $\Delta T_s = 0°$

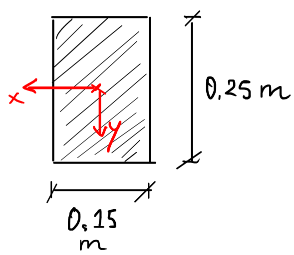
$q = 20$ kN/m , $\delta = 1$ cm

$L = 4$ m ;

$E = 206$ GPa

$\alpha = 1 \times 10^{-5}$ /°C

Cross – section



1 Degree of static indeterminacy :

$$DSI = ED + ID = (\underbrace{+5}_{\text{number of reactions}} \overbrace{-3}^{\text{number of dof}}) + \underbrace{(-1)}_{\text{hinge}} = +1$$ hyperstatic 1 time

OR

$DOCs = \underset{(A)}{+3} + \underset{(C)}{2} + \underset{(E)}{2} = 7$

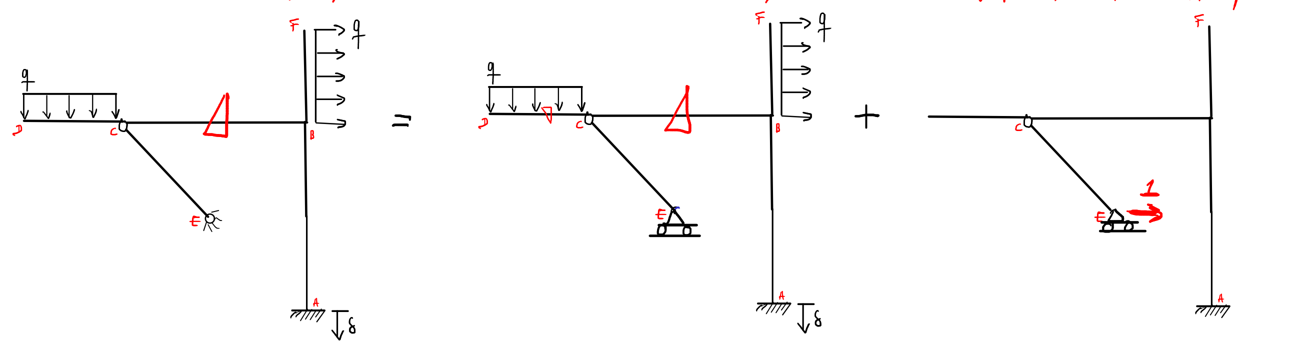
$DOFs = \underbrace{2}_{\text{number of bars}} \times \underbrace{3}_{\text{number of dofs}} = 6$

$DOCs > DOFs \Rightarrow 7 - 6 = +1$ hyperstatic 1 time

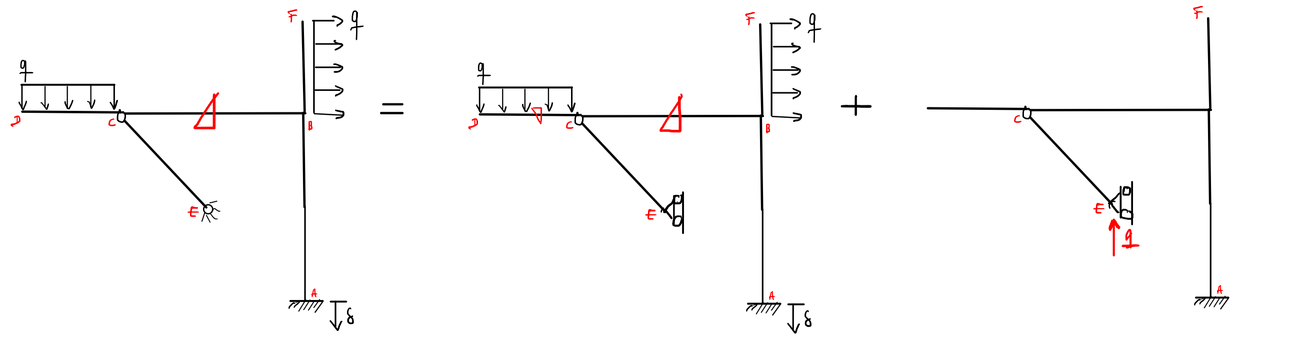
2 Definition of the BASE SYSTEM and virtual SYSTEM (superposition of effects)

REAL (HYPERSTATIC) = BASE (ISOSTATIC) + VIRTUAL (ISOSTATIC)

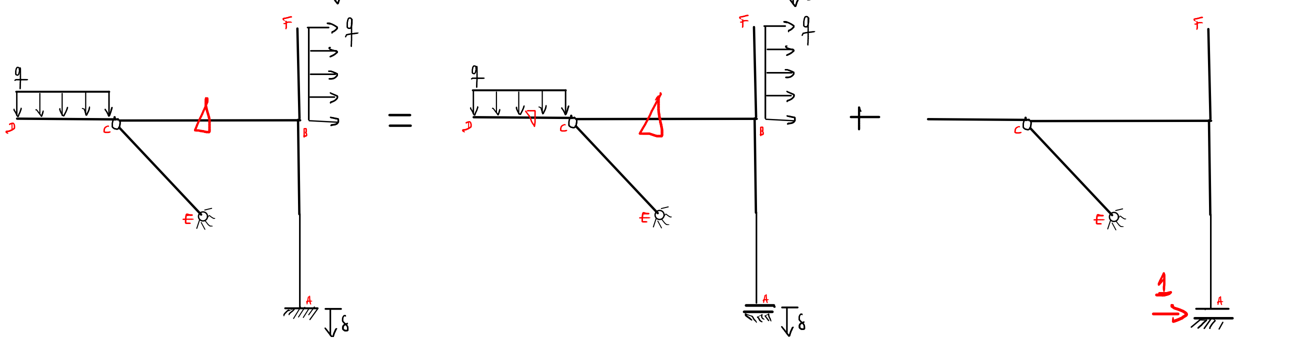
Option 1



Option 2



Option 3



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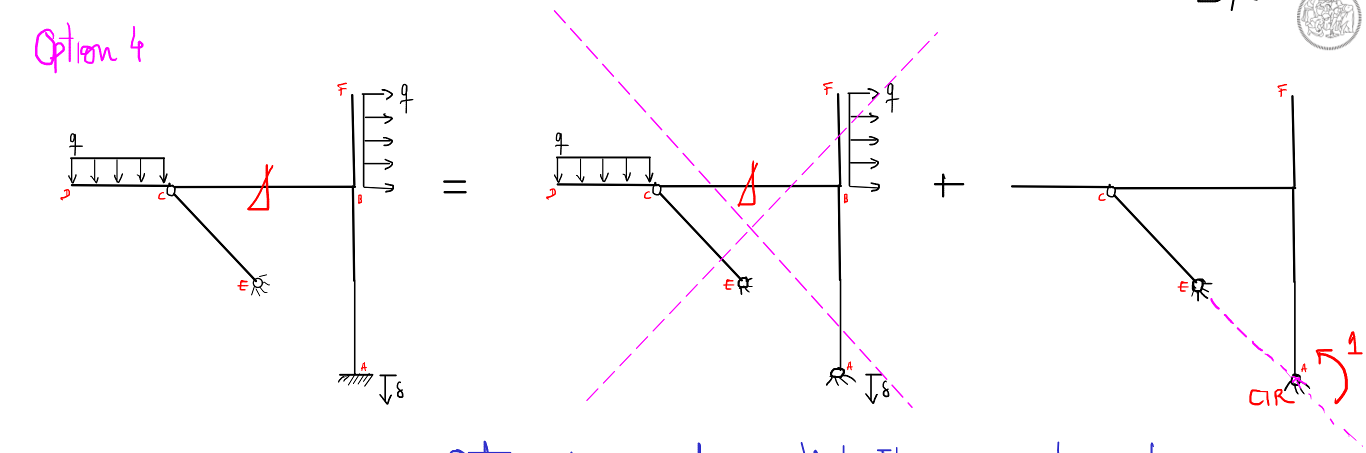
Option 4



Option 4 is not possible! It is a mechanism!

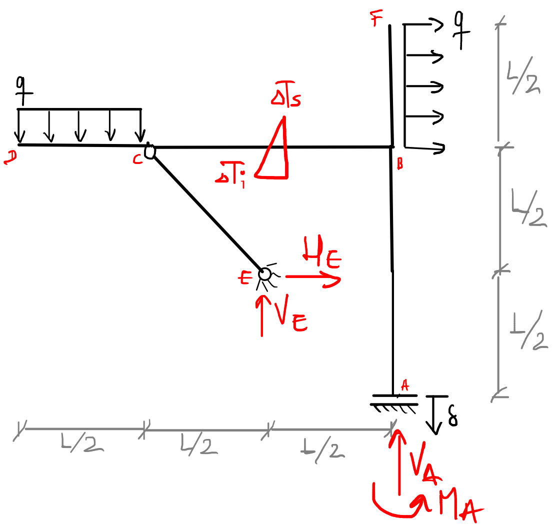
Let us select option $\boxed{3}$

$\boxed{3}$ PRINCIPAL OF VIRTUAL WORK ⇒ COMPATIBILITY CONDITION

I. The hyperstatic unknown is a horizontal Reaction in A;
II. this means that the compatibility equation is written in terms of displacements;

$$\underset{\text{BASE ROTATION}}{u_{Ax}^{0}} + \underset{\text{VIRTUAL ROTATION}}{X\,u_{Ax}^{1}} = \underset{\text{REAL ROTATION}}{0}$$

$\boxed{4}$ STUDY OF THE BASE SYSTEM, 0



GLOBAL EQUILIBRIUM (Reactions) (4 unknowns ⇒ 4 EQUATIONS)

$$
\left\{
\begin{aligned}
&\xrightarrow{+}\ \Sigma F_x = 0 && H_E + q\tfrac{L}{2} = 0 \Leftrightarrow H_E = -q\tfrac{L}{2}\ (\mathrm{kN}) \\
&+\uparrow\ \Sigma F_y = 0 && V_A + V_E - q\tfrac{L}{2} = 0 \\
&+\circlearrowleft\ \Sigma M_A = 0 && M_A - q\tfrac{L}{2}\cdot\left(\cancel{L}+\tfrac{L}{4}\right) + q\tfrac{L}{2}\cdot\left(\cancel{L}+\tfrac{L}{4}\right) - H_E\cdot\tfrac{L}{2} - V_E\cdot\tfrac{L}{2} = 0 \\
&+\circlearrowleft\ \Sigma M_C^{Right} = 0 && H_E\cdot\tfrac{L}{2} + V_E\cdot\tfrac{L}{2} = 0
\end{aligned}
\right.
$$

$H_E = -q\frac{L}{2} \Rightarrow$ correct $H_E = q\frac{L}{2}$ (kN) ←

$V_A + q\frac{L}{2} - q\frac{L}{2} = 0 \Leftrightarrow V_A = \emptyset$ $\quad V_A = \emptyset$

$M_A + q\frac{L}{2}\cdot\frac{L}{2} - q\frac{L}{2}\cdot\frac{L}{2} = 0 \Leftrightarrow M_A = \emptyset$ $\quad M_A = \emptyset$

$-q\frac{L}{2}\cdot\frac{L}{2} + V_E\cdot\frac{L}{2} = 0 \Leftrightarrow V_E = q\frac{L}{2}$ (kN) $\quad V_E = q\frac{L}{2}$ (kN) ↑

$\boxed{4.1}$ Internal Actions

Left (⇐ ↑ $\boxed{+}$ ↓ ⇒) Right

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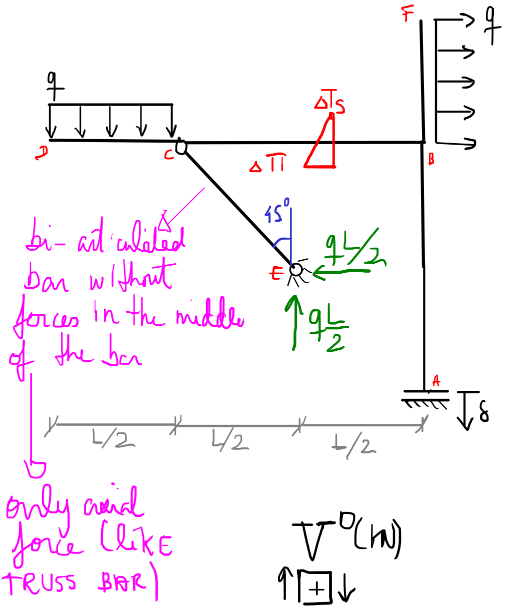
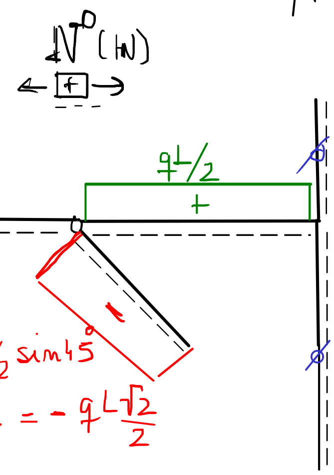
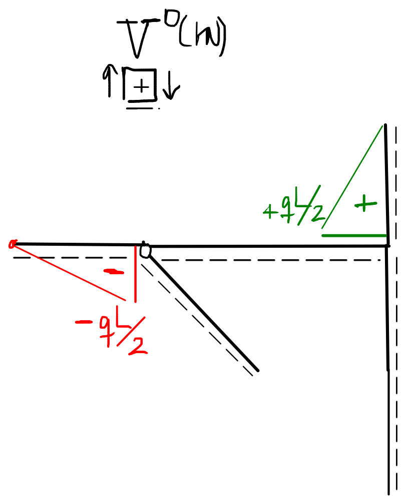
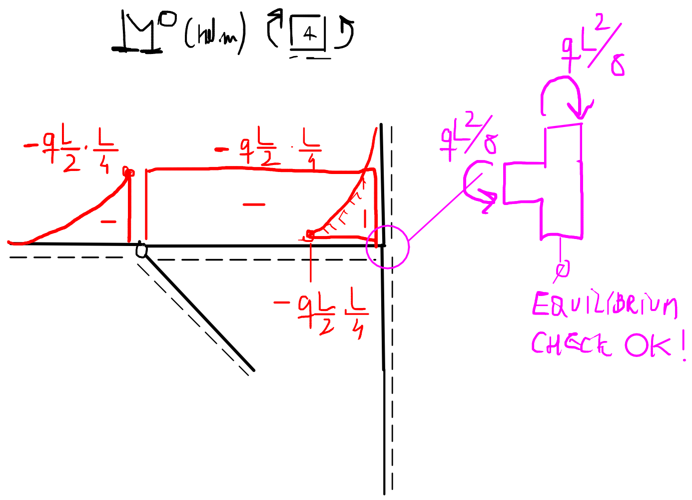


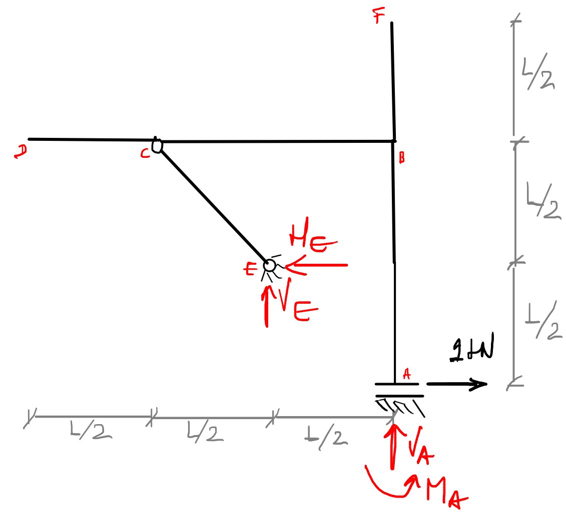


$$= -\frac{qL}{2}\cdot\cos 45 - \frac{qL}{2}\sin 45$$
$$= -2\frac{qL}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{qL\sqrt{2}}{2}$$





5 STUDY OF THE VIRTUAL SYSTEM



$\xrightarrow{+} \sum F_x = 0 \Leftrightarrow -H_E + 1 = 0 \Leftrightarrow H_E = 1\,kN\ (\leftarrow)$ (1)

$+\uparrow \sum F_y = 0 \Leftrightarrow V_A + V_E = 0 \Leftrightarrow V_A = -V_E \Leftrightarrow V_A = -1\,kN$ (3)

correct the sense

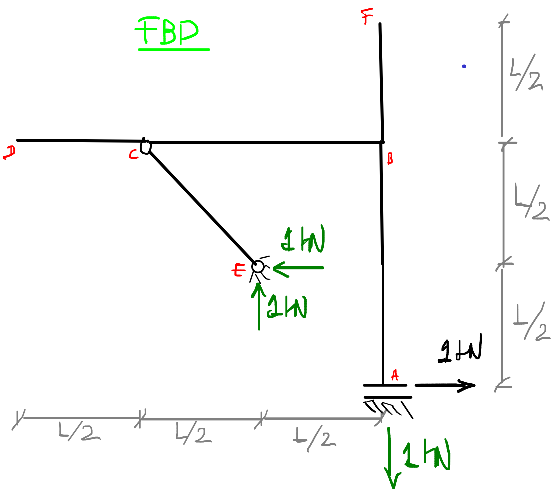
$\circlearrowleft_+ \sum M_A = 0 \Leftrightarrow M_A + H_E \cdot \frac{L}{2} - V_E \cdot \frac{L}{2} = 0 \Leftrightarrow M_A = \emptyset$ (4)

$\circlearrowleft_+ \sum M_C^{Right} = 0 \Leftrightarrow -H_E \cdot \frac{L}{2} + V_E \cdot \frac{L}{2} = 0 \Leftrightarrow V_E = H_E \Leftrightarrow V_E = 1\,kN\ (\uparrow)$ (2)

Correct the sense of V_A: $V_A = 1\,kN\ (\downarrow)$

5.1 Internal Actions

FBD



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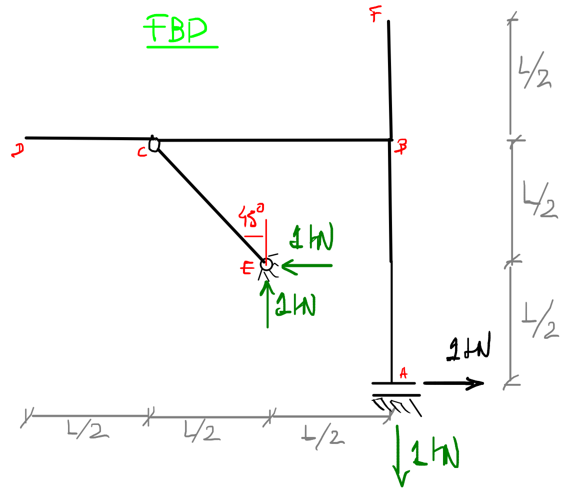
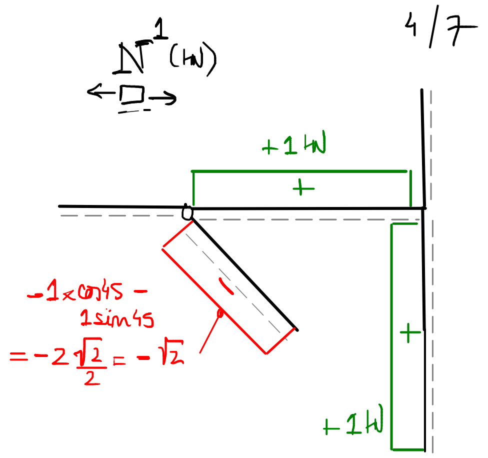
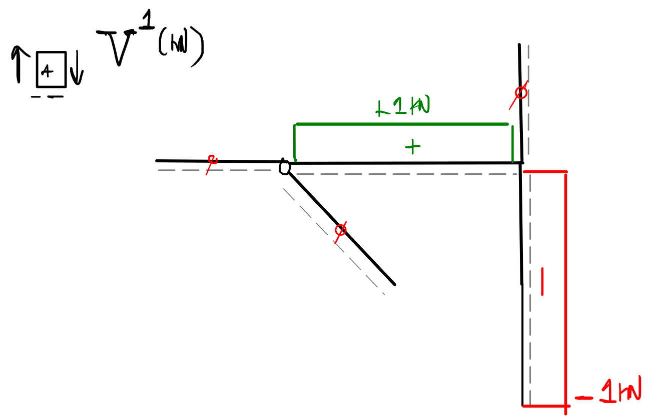
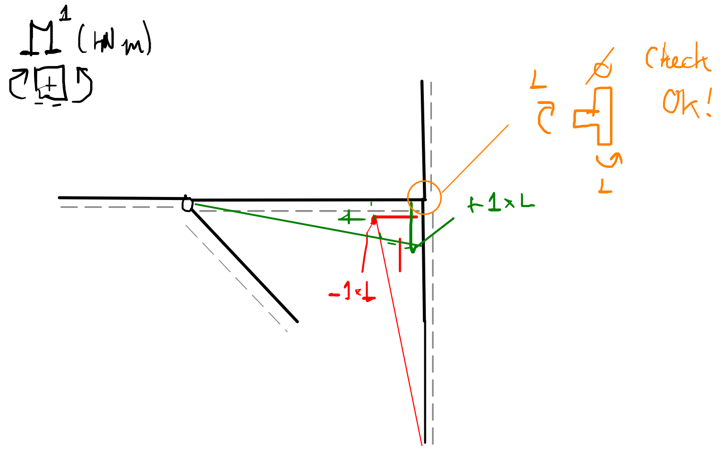








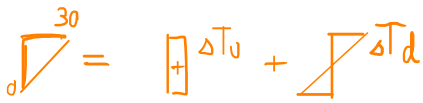
6 SOLVING THE COMPATIBILITY CONDITION

$$u^0_{A_x} + X\, u^1_{A_x} = 0$$

BASE ROTATION — VIRTUAL ROTATION — REAL ROTATION

BASE SYSTEM 0 ~ VIRTUAL SYSTEM 1

6.1 - SOLVING $u^0_{A_x}$



$W_{ext} = W_{int}$ ⇔

ignore — temperature effects

$$\Leftrightarrow \underbrace{1\times u^0_{A_x}}_{①} + \underbrace{\sum_i R^1_i\,\delta_i}_{\text{settlements}} = \underbrace{\int_l \frac{N^1N^0}{EA}dx}_{②} + \cancel{\int_l \frac{V_1V_0}{GA}dx} + \underbrace{\int_l \frac{M^1M^0}{EI}dx}_{③} + \underbrace{\int_l N^1\alpha\Delta T_0\,dx}_{④} + \underbrace{\int_l M^1\alpha\frac{\Delta T_d}{h}dx}_{⑤}$$

settlements. Only in A

①

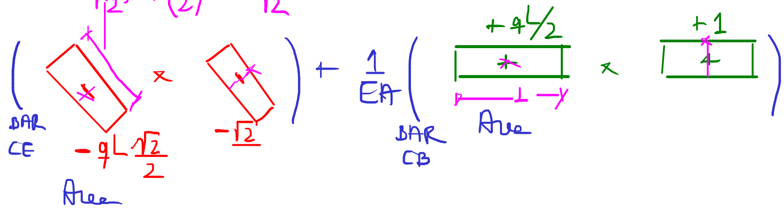
$$W_{ext} = u^0_{A_x} + \underbrace{(1\times\delta)}$$

⊕ because Reaction in A in the virtual system has the same sense of the settlement

②

$$\sqrt{\left(\frac{L}{2}\right)^2+\left(\frac{L}{2}\right)^2} = \frac{L}{\sqrt2}$$

$$\int_l \frac{N^1N^0}{EA}dx = \frac{1}{EA}\left(\ \right) + \frac{1}{EA}\left(\ \right)$$



$$= \frac{1}{EA}\left(-\frac{qL\sqrt2}{2}\cdot\frac{L}{\sqrt2}\cdot(-\sqrt2)\right) + \frac{1}{EA}\left(\frac{qL}{2}\cdot L\times(+1)\right) = \frac{1}{EA}\left(+\frac{qL^2}{2}\sqrt2 + \frac{qL^2}{2}\right)$$

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③

$$\int_\ell \frac{M^1 M^0}{EI}\,dx = \frac{1}{EI}\left(\underset{\text{BAR CB}}{} \; \left[-\frac{qL^2}{8}\right]_{\text{Area},\,L} \times \left[+L\right]_{\frac{L}{2}} \right) = \frac{1}{EI}\left(-\frac{qL^2}{8}\times L \times \frac{L}{2}\right) = -\frac{qL^4}{16EI}$$

④ temperature effects

$$\int_\ell N^1 \alpha \Delta T_0\,dx = \left(\underset{\text{Bar CB}}{} \left[+1\right]_{\text{Area},\,L} \times \alpha \times \frac{\Delta T_s + \Delta T_i}{2} \right) = 1 \times L \times \alpha \times \frac{\Delta T_s + \Delta T_i}{2}$$

⑤

$$\int_\ell M^1 \alpha \frac{\Delta T_d}{h}\,dx = \left(\underset{\text{BAR CB}}{} \left[+L\right]_{L} \times \alpha \times \frac{\Delta T_i - \Delta T_s}{h} \right) = \frac{L^2}{2}\cdot\alpha\cdot\frac{\Delta T_i - \Delta T_s}{h}$$

Replacing,

$$u^0_{Ax} + (1\times\delta) = \overbrace{\frac{1}{EA}\left(+\frac{qL^2}{2}\sqrt{2} + \frac{qL^2}{2}\right)}^{N} \; \overbrace{-\frac{qL^4}{16EI}}^{M} \; \overbrace{+1\times L\times\alpha\times\frac{\Delta T_s+\Delta T_i}{2}}^{\Delta T_U} \; \overbrace{+\frac{L^2}{2}\cdot\alpha\cdot\frac{\Delta T_i-\Delta T_s}{h}}^{\Delta T_D}$$

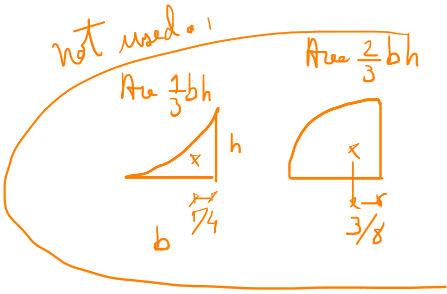
$$\Leftrightarrow u^0_{Ax} = \overbrace{\frac{1}{EA}\left(+\frac{qL^2}{2}\sqrt{2} + \frac{qL^2}{2}\right)}^{N} \; \overbrace{-\frac{qL^4}{16EI}}^{M} \; \overbrace{+1\times L\times\alpha\times\frac{\Delta T_s+\Delta T_i}{2}}^{\Delta T_U} \; \overbrace{+\frac{L^2}{2}\cdot\alpha\cdot\frac{\Delta T_i-\Delta T_s}{h}}^{\Delta T_D} \; \overbrace{-(1\times\delta)}^{\text{settlement}}$$

6.2 – SOLVING u^1_{Ax}

$W_{ext} = W_{int}$ (two times the virtual system)

$$\Leftrightarrow 1 \times u^1_{Ax} = \underbrace{\int_\ell \frac{N^1 N^1}{EA}\,dx}_{①} + \underbrace{\int_\ell \frac{M^1 M^1}{EI}\,dx}_{②}$$

not used.



①

$$\int_\ell \frac{N^1 N^1}{EA}\,dx = \frac{1}{EA}\left(\underset{\text{BAR AB}}{} \left[+1\right]_{\text{Area},\,L} \times \left[+1\right] \right) + \frac{1}{EA}\left(\underset{\text{BAR BC}}{} \left[+1\,kN\right]_{\text{Area},\,L} \times \left[+1\,kN\right] \right) + \frac{1}{EA}\left(\underset{\text{CE}}{} \left[-\sqrt{2}\right]_{\text{Area},\,L/\sqrt{2}} \times \left[-\sqrt{2}\right] \right)$$

$$= \frac{1}{EA}\left(L + L + \left(-\sqrt{2}\times\frac{L}{\sqrt{2}}\right)\times\left(-\sqrt{2}\right)\right) = \frac{1}{EA}\left(2L + \sqrt{2}L\right)$$

②

$$\int_\ell \frac{M^1 M^1}{EI}\,dx = \frac{1}{EI}\left(\underset{\text{AB}}{} \left[-L\right]_{\text{Area},\,L} \times \left[-L\right]_{\frac{2}{3}(-L)} \right) + \frac{1}{EI}\left(\underset{\text{BC}}{} \left[+L\right]_{\text{Area},\,L} \times \left[+L\right]_{\frac{2}{3}(L)} \right) = \frac{2}{EI}\left(\frac{L^2}{2}\cdot\frac{2}{3}L\right) = \frac{2}{3}\frac{L^3}{EI}$$

Replacing,

$$\boxed{u^1_{Ax} = \frac{1}{EA}\left(2L + \sqrt{2}L\right) + \frac{1}{EI}\left(\frac{2}{3}L^3\right)}$$

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6.3 Hyperstatic Unknown X

$$u^0_{A_x} + X u^1_{A_x} = 0$$

$$\left[\frac{1}{EA}\left(+\frac{qL^2}{2}\sqrt{2} + \frac{qL^2}{2}\right) - \frac{qL^4}{16EI} + 1\times L\times\alpha\times\frac{\Delta T_s+\Delta T_i}{2} + \frac{L^2}{2}\cdot\alpha\cdot\frac{\Delta T_i-\Delta T_s}{h} - (1\times\delta)\right] + \dots$$

$$\dots X\cdot\left[\frac{1}{EA}\left(2L+\sqrt{2}L\right) + \frac{1}{EI}\left(\frac{2}{3}L^3\right)\right]=0$$

knowing that $\delta = 1\times10^{-2}\,m$, $L = 4\,m$, $E = 206\times10^6\,kPa$, $q = 20\,kN/m$, $T_i = 30°C$, $T_s = 0°C$

And calculating: $A = 0.15\times0.25 = 0.0375\,m^2$; $I_{xx} = \frac{bh^3}{12} = \frac{0.15\times0.25^3}{12} = 19.53125\times10^{-5}\,m^4$

Replacing,

$$X = 7.25\ kN$$

7 INTERNAL ACTIONS FOR THE HYPERSTRUCTURE STRUCTURE

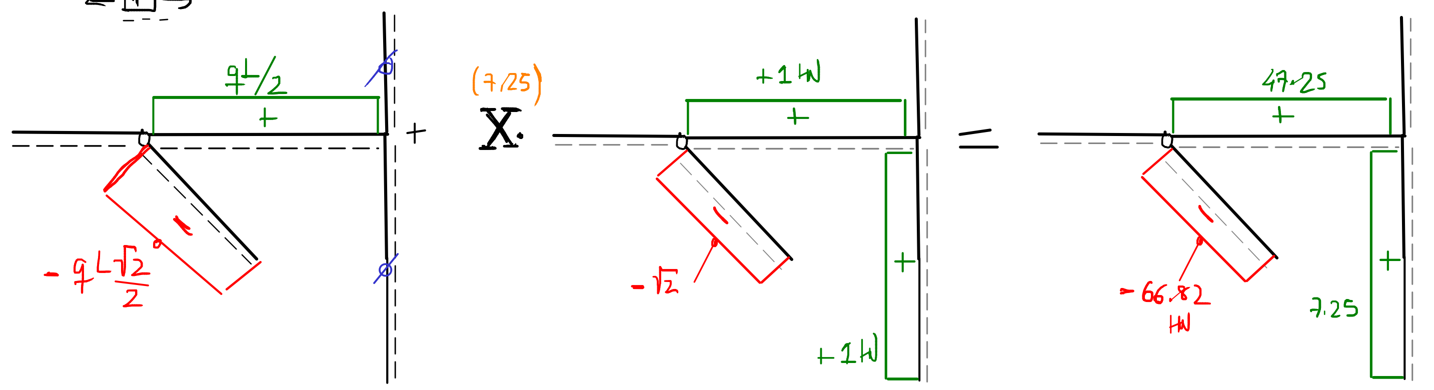
$N^R = N^0 + N^1\cdot X$

$V^R = V^0 + V^1\cdot X$

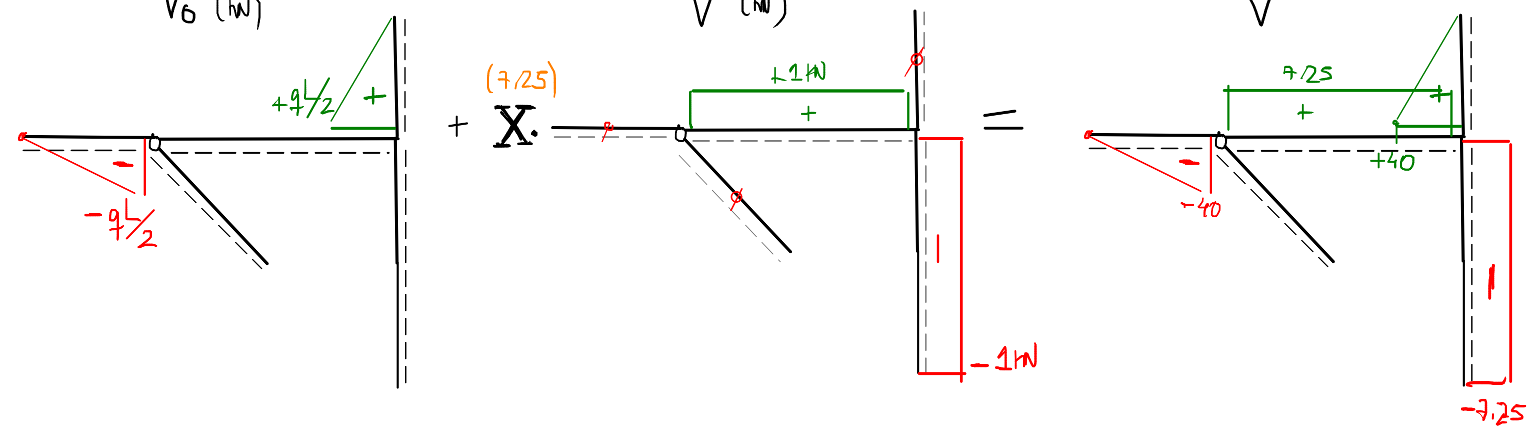
$M^R = M^0 + M^1\cdot X$

$\Rightarrow$ Making use of the compatibility equation

N^0 (kN) $\quad$ N^1 $\quad$ N^R (kN)



V_0 (kN) $\quad$ V^1 (kN) $\quad$ V^R



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